JOURNAL OF FUNCTIONAL ANALYSIS 34, 29–53 (1979)

Restrictions and Expansions of Holomorphic Representations

HANS PLESNER JAKOBSEN*

Brandeis University, Waltham, Massachusetts

AND

MICHELE VERGNE†

Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Communicated by the Editors

Received May 1, 1978

We compute tensor products of representations of the holomorphic discrete series of a Lie group G, or restrictions to some subgroup G'. A detailed study is done for the case of the conformal group $O(4, 2)$.

INTRODUCTION

Let D_n be a homogeneous bounded domain in $\mathbb{C}^n$ containing the origin 0. Let G_n be the group of holomorphic transformations of D_n and let us consider a representation T_n of G inside a Hilbert space H_n of holomorphic functions on D_n.

As a simple example, suppose that $D_{n-1} = \{D_n \cap z_n = 0\}$ is a homogeneous bounded domain in $\mathbb{C}^{n-1}$ for a subgroup G_{n-1} of G_n. We consider the restriction of the representation T_n to G_{n-1}. Clearly, as the Hilbert space H_n consists of holomorphic functions, the restriction map $R_0: f(z_1, z_2, ..., z_{n-1}, z_n) \to f(z_1, z_2, ..., z_{n-1}, 0)$ intertwines the representation T_n with a representation T_{n-1} of G_{n-1} inside a space of holomorphic functions on D_{n-1}. The kernel of R_0 is the subspace of holomorphic functions in H_n which vanish where $z_n = 0$. Similarly, the maps $R^p = ((\partial/\partial z_n)^p \cdot f)\,|_{z_n=0}$ are well defined, therefore it is natural, in order to calculate $T_n\,|_{G_{n-1}}$, to expand the functions f in H_n in Taylor series with respect to the variable z_n.

Let $D = G/K$ be a Hermitian symmetric space. We will consider a representation T of G of the holomorphic discrete series. The preceding simple idea of taking normal derivatives gives us the decomposition of the restriction of T to any

* Supported in part by NSF Grant MCS76-11312.
† Supported in part by NSF Grant MCS7602762.

0022-1236/79/100029–25$02.00/0

subgroup G' of G for which G'/K' is a complex submanifold of G/K (Proposition 2.5).

The tensor product of two representations of the holomorphic discrete series is included in this study, as it corresponds to the diagonal submanifold $D' = (z, z) \subset D \times D$ (Corollary 2.6). The idea of considering filtrations according to the order of vanishing along submanifolds is due to Martens [14], who used it to compute the characters of the holomorphic discrete series. We will refer to it as the Martens method.

Sections 3 and 4 deal with matters relevant to theoretical physics. The problems arose in connection with the extension to microphysics of the macrophysical theory of Segal [11]. We have benefited from discussions with B. Orsted, B. Speh, and I. Segal concerning these problems. The physical significance of the restriction question for representations of $O(4, 2)$ to $O(3, 2)$ is indicated in part in our joint note, (Jacobsen *et al.* [6]) and in part by Segal [11]. The scalar case has been treated by another method in particularly explicit form by Orsted [9], who also considered the analogous question for the action of $O(p, q)$ on $S^{p-1} \times S^{q-1}$. We believe that Martens method shed some light on these matters, for the particular case of the group $O(2, n)$.

In Section 3, we study the modules of K-finite functions on the Minkowski-space $\mathbb{R}^{1+n}$ obtained by taking boundary values of holomorphic functions on the associated homogeneous tube domain $\mathbb{R}^{1+n} + iC^+$, where C^+ is the solid light cone. These modules are the "positive-energy" subspaces of some of the classical representation of the group $O(2, n+1)$ acting by conformal transformations on the Minkowski space. We give the decomposition of the positive energy subspace under the Lie algebra of $O(2, n)$. In particular, it is multiplicity free. We also obtain information about the Fourier transforms of the modules.

In Section 4, we study the decomposition of tensor products of unitary representations of $SU(2, 2)$ in reproducing kernel Hilbert spaces of holomorphic functions. Our main interest is in the case in which at least one factor in the tensor product lives in a solution space to a "Mass-zero" equation. Some of the results apply equally well to $SU(n, n)$ and $Mp(n, \mathbb{R})$; however, to complete the analysis some concrete computations are needed, and we only do these for $SU(2, 2)$. We finish this paragraph by observing that certain differential equations appear naturally in the decomposition of such tensor products.

We wish to thank M. Kashiwara, E. Gutkin, B. Orsted, I. E. Segal, and B. Speh for helpful discussions.

1. The General Case; Modules

Let G be a Lie group with Lie algebra g. Let G_1 and G_2 be connected subgroups of G with Lie algebras g_1 and g_2. Let τ_1 be a representation of g_1 in a finite-dimensional vector space V_{τ_1}, let $\mathscr{U}(g)$ be the enveloping algebra of g and let

$$M = M(g, g_1, V_{\tau_1}) = \mathscr{U}(g) \bigotimes_{\mathscr{U}(g_1)} V_{\tau_1} \tag{1.1}$$

be the induced module.

We consider M as a g_2-module and let $\mathscr{U}_0(g) \subseteq \cdots \subseteq \mathscr{U}_n(g) \subseteq \cdots$ be the canonical filtration of $\mathscr{U}(g)$. Then the modules

$$M_n = \mathscr{U}(g_2)\, \mathscr{U}_n(g)(1 \otimes V_{\tau_1}) \tag{1.2}$$

clearly form an increasing sequence of g_2-modules.

For a vector space V we denote by $S(V) = \oplus^n S^n(V)$ the symmetric algebra of V with its canonical gradation. We consider the natural action of $g_1 \cap g_2$ on $g/g_1 + g_2$; with this $S^n(g/g_1 + g_2)$ is a $g_1 \cap g_2$ module.

It is easy to check that the $\mathscr{U}(g_2)$ module M_n/M_{n-1} is isomorphic to

$$\mathscr{U}(g_2) \bigotimes_{\mathscr{U}(g_1 \cap g_2)} (S^n(g/g_1 + g_2) \otimes V_{\tau_1}).$$

We assume that we are given a representation τ_1' of G_1 in the dual space V_{τ_1}' of V_{τ_1}, whose differential, also denoted by τ_1', equals the contragredient representation of τ_1. The space $\mathscr{O}(G, G_1, V_{\tau_1}')$ of analytic functions $\varphi\colon G \to V_{\tau_1}'$ for which, for all $(g, g_1) \in G \times G_1$,

$$\varphi(gg_1) = \tau_1'(g_1)^{-1}\varphi(g) \tag{1.3}$$

can be identified with the space of analytic sections of the bundle

$$G \times_{G_1} V_{\tau_1}' \to G/G_1. \tag{1.4}$$

G acts on $\mathscr{O}(G, G_1, V_{\tau_1}')$ by left translation.

For $x \in g$ we denote by $r(x)$ the differential operator which acts on C^∞-functions φ from G to V_{τ_1}' by

$$(r(x)\varphi)(g) = \frac{d}{dt}\, \varphi(g \exp tx)\,|_{t=0}, \tag{1.5}$$

and extend r to $\mathscr{U}(g^{\mathbf{C}})$. Clearly, if $\varphi \in \mathscr{O}(G, G_1, V_{\tau_1}')$ and if $x \in g_1$,

$$(r(x)\varphi)(g) = -\tau_1'(x)\, \varphi(g). \tag{1.6}$$

For $u \in \mathscr{U}(g^{\mathbf{C}})$, $v \in V_{\tau_1}$, and $\varphi \in \mathscr{O}(G, G_1, V_{\tau_1}')$ we now define

$$(u \otimes v, \varphi)(g) = \langle v, (r(u)\varphi)(g)\rangle, \tag{1.7}$$

and observe that this only depends upon u and v through the image ξ of $u \otimes v$ in $M(\mathfrak{g}, \mathfrak{g}_1, V_{\tau_1})$. The corresponding function is denoted by $(\xi, \varphi)(g)$, and we define

$$\langle \xi, \varphi \rangle = (\xi, \varphi)(e). \tag{1.8}$$

If G is connected and if, for some φ in $\mathcal{Cl}(G, G_1, V'_{\tau_1})$, $\langle \xi, \varphi \rangle = 0$ for all ξ in $M(\mathfrak{g}, \mathfrak{g}_1, V_{\tau_1})$, then φ is zero, since it is assumed to be analytic.

The group G_2 acts on G/G_1, and since

$$G_2 \cdot \underline{e} \simeq G_2/G_1 \cap G_2, \tag{1.9}$$

at the point $\underline{e}$, the tangent space to the submanifold $G_2 \cdot \underline{e}$ of G/G_1 is $\mathfrak{g}_1 + \mathfrak{g}_2/\mathfrak{g}_1$, and hence the complexified normal space is $(\mathfrak{g}/\mathfrak{g}_1 + \mathfrak{g}_2)^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}/\mathfrak{g}_1^{\mathbb{C}} + \mathfrak{g}_2^{\mathbb{C}}$.

The restriction map R_0: $\mathcal{Cl}(G, G_1, V'_{\tau_1}) \to \mathcal{Cl}(G_2, G_1 \cap G_2, V'_{\tau_1})$ defined by

$$(R_0\varphi)(g_2) = \varphi(g_2) \tag{1.10}$$

clearly commutes with the action of G_2. Moreover, if G_2 is connected, it is easy to see that

$$R_0\varphi = 0 \Leftrightarrow \forall \xi \in M_0 = \mathcal{U}(\mathfrak{g}_2)(1 \otimes V_{\tau_1}) : \langle \xi, \varphi \rangle = 0. \tag{1.11}$$

Let $\mathcal{Cl}_0$ be the kernel of R_0:

$$\mathcal{Cl}_0 = \{\varphi \in \mathcal{Cl}(G, G_1, V'_{\tau_1}) \mid R_0\varphi = 0\}. \tag{1.12}$$

We can then define a map R_1 from $\mathcal{Cl}_0$ to $\mathcal{Cl}(G_2, G_1 \cap G_2, (V_{\tau_1} \otimes (\mathfrak{g}^{\mathbb{C}}/\mathfrak{g}_1^{\mathbb{C}} + \mathfrak{g}_2^{\mathbb{C}}))')$ by

$$\langle (R_1\varphi)(g_2), v \otimes \dot{x} \rangle = \langle (r(x)\varphi)(g_2), v \rangle. \tag{1.13}$$

Here, $v \in V'_{\tau_1}$, $\dot{x} \in (\mathfrak{g}/\mathfrak{g}_1 + \mathfrak{g}_2)^{\mathbb{C}}$, and $x \in \mathfrak{g}^{\mathbb{C}}$ is chosen to be such that the equivalence class of x modulo $\mathfrak{g}_1^{\mathbb{C}} + \mathfrak{g}_2^{\mathbb{C}}$ is $\dot{x}$.

That R_1 is a well-defined map with values in $\mathcal{Cl}(G_2, G_1 \cap G_2, (V_{\tau_1} \otimes (\mathfrak{g}^{\mathbb{C}}/\mathfrak{g}_1^{\mathbb{C}} + \mathfrak{g}_2^{\mathbb{C}}))')$ follows by noting that if $R_0\varphi$ is zero, then so is $R_0 r(y)\varphi$ for any y in $\mathfrak{g}_1^{\mathbb{C}} + \mathfrak{g}_2^{\mathbb{C}}$. Also, if $g_0 \in G_1 \cap G_2$, then $\mathrm{Ad}\underline{h}(g_0)$ leaves $\mathfrak{g}_1 + \mathfrak{g}_2$ invariant.

R_1 can be interpreted as the map which associates to each function φ which vanishes on $G_2 \cdot \underline{e}$ the derivatives of φ along the "normal directions" to $G_2 \cdot \underline{e}$.

This procedure can of course be continued. We let

$$\mathcal{Cl}_{n-1} = \{\varphi \mid \forall \xi \in M_{n-1} : R_0(\xi, \varphi) = 0\} \tag{1.14}$$

and define

$$R_n : \mathcal{Cl}_{n-1} \to \mathcal{Cl}(G_2, G_1 \cap G_2, (V_{\tau_1} \otimes S^n(\mathfrak{g}^{\mathbb{C}}/\mathfrak{g}_1^{\mathbb{C}} + \mathfrak{g}_2^{\mathbb{C}}))') \tag{1.15}$$

by

$$\langle (R_n\varphi)(g_2), v \otimes \dot{b}\rangle = \langle (r(b)\varphi)(g_2), v\rangle \tag{1.16}$$

where $b \in \mathscr{U}_n(g^{\mathbf{C}})$ is such that its image under the map $\mathscr{U}_n(g^{\mathbf{C}}) \to S^n(g^{\mathbf{C}}) \to S^n(g^{\mathbf{C}}/g_1{}^{\mathbf{C}} + g_2{}^{\mathbf{C}})$ is $\dot{b}$.

We will apply this line of thought to the study of certain modules of holomorphic functions on Hermitian symmetric spaces.

2. The Case of a Hermitian Symmetric Space

Let g be a semisimple Lie algebra and let $g = k \oplus p$ be a Cartan decomposition of g. We assume that k has a nontrivial center and choose a Cartan subalgebra $h^{\mathbf{C}}$ contained in $k^{\mathbf{C}}$, and a system of positive roots Δ^+ of $(g^{\mathbf{C}}, h^{\mathbf{C}})$ such that

$$g^{\mathbf{C}} = k^{\mathbf{C}} \oplus p^+ \oplus p^- \tag{2.1}$$

where $p^+ \subseteq \sum_{\alpha\in\Delta^+} g^\alpha$, $p^- \subseteq \sum_{-\alpha\in\Delta^+} g^\alpha$, the spaces p^+ and p^- are abelian subalgebras of $g^{\mathbf{C}}$, and

$$[k^{\mathbf{C}}, p^+] \subseteq p^+; \qquad [k^{\mathbf{C}}, p^-] \subseteq p^-. \tag{2.2}$$

We also define

$$\Delta_{K^+} = \{\alpha \in \Delta^+ \mid g^\alpha \subseteq k^{\mathbf{C}}\}$$

and

$$\Delta_{P^+} = \{\alpha \in \Delta^+ \mid g^\alpha \subseteq p^+\}. \tag{2.3}$$

The center ζ of k is one dimensional. We choose $z \in \zeta$ such that

$$\forall x \in p^+ \colon [z, x] = ix. \tag{2.4}$$

Let $G^{\mathbf{C}}$ be the simply connected Lie group with Lie algebra $g^{\mathbf{C}}$, and let $K^{\mathbf{C}}$, G, and K be the connected subgroups corresponding to $k^{\mathbf{C}}$, g, and k. The space $D = G/K$ then has the structure of a Hermitian symmetric space, and the space of holomorphic functions on D can be identified with the set of analytic functions φ on G/K for which $r(x)\varphi = 0$ for all x in p^-. We finally recall that any element $g \in G$ uniquely can be written as

$$g = \exp X_+(g) \cdot k(g) \cdot \exp X_-(g) \tag{2.5}$$

where $X_+(g) \in p^+$, $X_-(g) \in p^-$, and $k(g) \in K^{\mathbf{C}}$, and that the map $g \to X_+(g)$ is an isomorphism of D onto a bounded domain $D(p^+)$ contained in p^+ [3, 4].

We shall use this version of D, and observe that the action of $k \in K$ on $D(p^+)$ is given by $\mathrm{Ad}(k)$. In particular, the group $\exp tz$ acts by

$$(\exp tz)\xi = e^{it} \cdot \xi \qquad \text{for} \quad \xi \in D = D(p^+). \tag{2.6}$$

Let τ be a finite-dimensional irreducible unitary representation of K in V. We let

$$\mathcal{O}(G, K, V_\tau) = \{\text{Analytic functions } \varphi\colon G \to V_\tau \mid \forall k \in K\colon \varphi(gk) = \tau(k)^{-1}\varphi(g), \text{ and } \forall x \in p^-\colon r(x)\varphi = 0\}. \tag{2.7}$$

The group G acts on $\mathcal{O}(G, K, V_\tau)$ by left translation. We let $\underline{k}^{\mathbf{C}}$ act on V_τ' by the extension of the contragredient representation of $\underline{k}$, and let p^- act trivially. As before we can then form the module

$$M(g^{\mathbf{C}}, \underline{k}^{\mathbf{C}} \oplus p^-, V_\tau') = \mathcal{U}(g^{\mathbf{C}}) \underset{\mathcal{U}(k^{\mathbf{C}} \oplus p^-)}{\otimes} V_\tau' \tag{2.8}$$

and we can define, for any $\varphi \in \mathcal{O}(G, K, V_\tau)$ and any $\xi \in M(g^{\mathbf{C}}, \underline{k} \oplus p^-, V_\tau)$, the function $(\xi, \varphi)(g)$.

If $\varphi \in \mathcal{O}(G, K, V_\tau)$, the function $(P\varphi)(g) = \tau(k(g))\,\varphi(g)$ is invariant under right translation by K and hence the map $\varphi \to P\varphi$ is an isomorphism between $\mathcal{O}(G, K, V_\tau)$ and $\mathcal{O}(D, V_t)$, the space of holomorphic functions from D to V_τ. The action of K on $\mathcal{O}(D, V_\tau)$ becomes

$$(k \cdot \psi)(\xi) = \tau(k) \cdot \psi(k^{-1}\xi) \tag{2.9}$$

and it is then clear from (2.6) that any finite-dimensional subspace of $\mathcal{O}(D, V_\tau)$, that transforms according to an irreducible representation of K, consists of restrictions of polynomials on p^+ of a certain homogeneity to D. In other words: If $\mathcal{O}^f(G, K, V_\tau)$ denotes the subspace of $\mathcal{O}(G, K, V_\tau)$ spanned by the K-finite vectors, and if we identify the polynomials on p^+ with $S(p^-)$ via the Killing form, then we have that as K-modules $S(p^-)$ via the Killing form, then we have that as K-modules, $\mathcal{O}^f(G, K, V_\tau) = S(p^-) \otimes V_\tau$.

Similar to (1.5) we define $l(x)$ for $x \in g$, by

$$(l(x)\varphi)(g) = \frac{d}{dt}\,\varphi(\exp - txg)\,|_{t=0}, \tag{2.10}$$

and extend l to $\mathcal{U}(g^{\mathbf{C}})$.

If we identify $\mathcal{O}^f(G, K, V_\tau)$ with the space of polynomials on p^+ with values in V_τ, and if, for any C^∞-function f on p^+,

$$(\delta(v_0)f)(v) = \frac{d}{ds} f(v + sv_0)\,|_{s=0} \tag{2.11}$$

for v_0 and v in $\mathfrak{p}^+$, the action of $l(x)$ for $x \in \mathfrak{g}^{\mathbf{C}}$ is given as follows:

$$\begin{aligned}
&\text{If } x \in \mathfrak{p}^+\colon (l(x)\,p)(v) = -(\delta(x)\,p)(v),\\
&\text{if } x \in \mathfrak{k}^{\mathbf{C}}\colon (l(x)\,p)(v) = d\tau(x)\,p(v) - (\delta([x, v])\,p)(v), \text{ and}\\
&\text{if } x \in \mathfrak{p}^-\colon (l(x)\,p)(v) = d\tau([x, v])\,p(v) - \tfrac{1}{2}(\delta([[x, v], v])\,p)(v).
\end{aligned} \tag{2.12}$$

For any v in V_τ the function

$$(\psi_\tau{}^v)(g) = \tau(k(g))^{-1} \cdot v \tag{2.13}$$

belongs to $\mathcal{O}^f(G, K, V_\tau)$ and satisfies $l(\mathfrak{p}^+)\,\psi_\tau{}^v = 0$. ($\psi_\tau{}^v$ is defined on the set $\exp \mathfrak{p}^+ K^{\mathbf{C}} \exp \mathfrak{p}^-$.) In particular, $P\psi_\tau{}^v$ is the constant function $\xi \to v$ on D.

We now define a module map $\Pi\colon \mathcal{U}(\mathfrak{g}^{\mathbf{C}}) \otimes_{\mathcal{U}(\mathfrak{k}^{\mathbf{C}} \oplus \mathfrak{p}^+)} V_\tau \to \mathcal{O}^f(G, K, V_\tau)$ by

$$\Pi(u \otimes v) = l(u)\,\psi_\tau{}^v, \tag{2.14}$$

and let $W(\tau)$ denote the image of $\mathcal{U}(\mathfrak{g}^{\mathbf{C}}) \otimes_{\mathcal{U}(\mathfrak{k}^{\mathbf{C}} \oplus \mathfrak{p}^+)} V_\tau$ under Π. It is easy to see that every $\mathfrak{g}$-submodule of $\mathcal{O}^f(G, K, V_\tau)$ contains $W(\tau)$. In particular, $W(\tau)$ is irreducible (and hence is *the* irreducible quotient of $\mathcal{U}(\mathfrak{g}^{\mathbf{C}}) \otimes_{\mathcal{U}(\mathfrak{k}^{\mathbf{C}} \oplus \mathfrak{p}^+)} V_\tau$) Since $\mathcal{O}^f(G, K, V_\tau)$ has the same decomposition under K as $\mathcal{U}(\mathfrak{g}^{\mathbf{C}}) \otimes_{\mathcal{U}(\mathfrak{k}^{\mathbf{C}} \oplus \mathfrak{p}^+)} V_\tau$, the preceding remark also shows that $\mathcal{O}^f(G, K, V_\tau)$ is isomorphic to $\mathcal{U}(\mathfrak{g}^{\mathbf{C}}) \otimes_{\mathcal{U}(\mathfrak{k}^{\mathbf{C}} \oplus \mathfrak{p}^+)} V_\tau$ if and only if the latter is irreducible.

Consider the following subsets of $\hat{K}$:

$$I = \left\{\tau \in \hat{K} \mid \mathcal{U}(\mathfrak{g}^{\mathbf{C}}) \bigotimes_{\mathcal{U}(\mathfrak{k}^{\mathbf{C}} \oplus \mathfrak{p}^+)} V_\tau \text{ is irreducible}\right\},$$

and

$$P = \{\tau \in \hat{K} \mid W(\tau) \text{ is unitarisable}\}. \tag{2.15}$$

We shall need the following result:

PROPOSITION 2.1 [3]. *Let $\tau \in P$. Then there exists a reproducing kernel Hilbert space $H(\tau)$ such that $W(\tau) \subseteq H(\tau) \subseteq \mathcal{O}(G, K, V_\tau)$ and $H(\tau)$ is the completion of $W(\tau)$. Moreover, if H is a Hilbert space contained in $\mathcal{O}(G, K, V_\tau)$ on which G acts unitarily, and if the evaluation map $\psi \to \psi(e)$ is a continuous map from H to $\mathbb{C}$ then $H = H(\tau)$. To be precise: As sets, H and $H(\tau)$ are equal, and the Hilbert space structures are proportional.*

If $\tau \in P$ we let T_τ denote the corresponding unitary representation of G in $H(\tau)$. Finally we observe that the set P is not known apart from some special cases [8, 10, 13].

Let $\mathfrak{g}_1$ be a semisimple subalgebra of $\mathfrak{g}$ such that $\mathfrak{g}_1 = \mathfrak{g}_1 \cap \mathfrak{k} \oplus \mathfrak{g}_1 \cap \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}_1$. Assume that $\mathfrak{p}_1{}^{\mathbf{C}} = \mathfrak{p}_1{}^{\mathbf{C}} \cap \mathfrak{p}^+ \oplus \mathfrak{p}_1{}^{\mathbf{C}} \cap \mathfrak{p}^-$. We let

G_1 and K_1 be the connected subgroups of G corresponding to $\underline{g}_1$ and $\underline{k}_1 = \underline{g}_1 \cap \underline{k}$, respectively. By the above assumptions $D_1 = G_1/K_1$ is a Hermitian symmetric space, and in the Harish-Chandra realization $D_1 \subset D \subset \underline{p}^+$, and $D_1 = D \cap \underline{p}_1^+$. As before we identify $\mathcal{O}^f(G, K, V_\tau)$ with the space of all polynomials on $\underline{p}^+$.

Since $\underline{p}_1^+ \subseteq \underline{p}^+$ we have that $\underline{p}^+ = \underline{p}_1^+ \oplus \underline{p}'^+$ where $\underline{p}'^+$ is the complement of $\underline{p}_1^+$ in $\underline{p}^+$. Corresponding to this we let $(x, y) = (x_1, \ldots, x_p, y_1, \ldots, y_q)$ be a set of coordinates on $\underline{p}^+$, with $x \in \underline{p}_1^+$, and $y \in \underline{p}'^+$.

Intuitively speaking we shall decompose G-modules of holomorphic functions on D under G_1 by expressing the corresponding functions $\varphi(x, y)$ by Taylor series;

$$\varphi(x, y) = \varphi(x, 0) + \sum_\alpha y^\alpha \varphi_\alpha(x).$$

In this spirit we define subspaces $\mathcal{O}_r$ of $\mathcal{O}^f(G, K, V_\tau)$ for $r = 0, 1, 2, \ldots$, by

$$\begin{aligned}\mathcal{O}_r = \operatorname{span}\{P_\alpha(x_1, \ldots, x_p)\, y_1^{\alpha_1} \cdots y_q^{\alpha_q} \mid P_\alpha &\text{ is a polynomial} \\ \text{on } \underline{p}^+ \text{ with values in } V_\tau, \text{ and } |\alpha| &= \alpha_1 + \cdots + \alpha_q \geqslant r\}. \quad (2.16)\end{aligned}$$

These subspaces $\mathcal{O}_r$ are in fact $\mathcal{U}(\underline{g}_1)$-modules as can be seen from (3.12): If $x \in \underline{p}_1^+$ then $\delta(x)$ only differentiates the polynomials through the x-variables, and if $x \in \underline{k}_1^{\mathbb{C}}$ $d\tau(x)$ is a linear operator that leaves both $\underline{p}_1^+$ and $\underline{p}'^+$ invariant whereas $\delta[x, v]$ splits into two terms, one of which is as above, and one which is a differential operator with first order polynomials in the y-variables as coefficients. Finally, if $x \in \underline{p}_1^-$, one can similarly see that $l(x)$ leaves $\mathcal{O}_r$ invariant.

We have $\underline{g}_1^{\mathbb{C}} = \underline{k}_1^{\mathbb{C}} \oplus \underline{p}_1^+ \oplus \underline{p}_1^-$ with $\underline{k}_1^{\mathbb{C}} = \underline{k}^{\mathbb{C}} \cap \underline{g}_1^{\mathbb{C}}$, $\underline{p}_1^+ = \underline{p}^+ \cap \underline{g}_1^{\mathbb{C}}$, and $\underline{p}_1^- = \underline{p}^- \cap \underline{g}_1^{\mathbb{C}}$. Hence, with $\underline{b}^{\mathbb{C}} = \underline{k}^{\mathbb{C}} \oplus \underline{p}^-$,

$$\underline{g}_1^{\mathbb{C}} + \underline{b}^{\mathbb{C}} = \underline{k}^{\mathbb{C}} \oplus \underline{p}_1^+ \oplus \underline{p}^-, \qquad \text{and} \qquad \underline{g}_1^{\mathbb{C}} \cap \underline{b}^{\mathbb{C}} = \underline{k}_1^{\mathbb{C}} \oplus \underline{p}_1^-. \quad (2.17)$$

Thus, $\underline{g}^{\mathbb{C}}/\underline{g}_1^{\mathbb{C}} + \underline{b}^{\mathbb{C}} \simeq \underline{p}^+/\underline{p}_1^+$ as representation spaces for $\underline{k}_1^{\mathbb{C}} \oplus \underline{p}_1^-$ where $\underline{p}_1^-$ acts trivially, and $\underline{k}_1^{\mathbb{C}}$ by the natural representation. $\underline{p}^+/\underline{p}_1^+$ can be viewed as the space of holomorphic vector fields normal to the domain $D_1 \subset D$ at the point $\underline{e}$.

The preceding analysis then leads to

Proposition 2.2. *$\mathcal{O}^f(G, K, V_\tau)$ has a filtration by $\mathcal{U}(\underline{g}_1)$-submodules $\mathcal{O}_r$ $(r = 0, 1, 2, \ldots)$. The quotients $\mathcal{O}_r/\mathcal{O}_{r+1}$ are canonically isomorphic to $\mathcal{O}^f(G_1, K_1, \tau \otimes (S^r(\underline{p}^+/\underline{p}_1^+))')$.*

Corollary 2.3. *Suppose that the representations μ of K_1 occurring in the decomposition of $\tau \otimes (S^r(\underline{p}^+/\underline{p}_1^+))'$, for every r, all are in the set I_1 for K_1 (cf. (2.15)). Then the $\mathcal{U}(\underline{g}_1)$-module $\mathcal{O}^f(G, K, V_\tau)$ splits into a direct sum.*

Proof. The functions $w = y_1^{a_1} \cdots y_q^{a_q} \cdot v$ $(v \in V_\tau)$ all satisfy: $l(p_1^+)w = 0$. Hence we can send $\mathscr{U}(g_1^{\mathbf{C}}) \otimes_{\mathscr{U}(k_1^{\mathbf{C}} \oplus p_1^+)} (V_\tau \otimes (S^r(p^+/p_1^+))')$ into $\mathscr{O}_r$. If the hypothesis of the corollary is satisfied this map is an isomorphism into a complement of $\mathscr{O}_{r+1}$.

We now turn to the case in which G acts unitarily in a Hilbert space $H(\tau) \subset \mathscr{O}(G, K, V_\tau)$ in which the point-evaluation maps are continuous, and consider the decomposition of $H(\tau)$ under G_1. We proceed as in Section 1 and let

$$H = H(\tau),$$

and

$$H_0 = \{\varphi \in H(\tau) \mid \varphi\,|_{G_1} = 0\}. \tag{2.18}$$

H_0 is the kernel of the map $R_0\colon H \to \mathscr{O}(G, K, V_\tau)$ defined by $(R_0\varphi)(g_1) = \varphi(g_1)$. (We shall continue to use the notation R_n for the maps corresponding to (1.15), even though the maps here in general are defined on different spaces.) Since H_0 clearly is a closed G_1-invariant subspace of H, H/H_0, and hence $R_0(H)$, has a canonical Hilbert space structure. It is easy to see that $R_0(H) = H(G_1, K_1, V_\tau)$ is a space of functions on G_1 in which the point-evaluation maps are continuous.

Let $\tau\,|_{K_1} = \bigoplus_{j\in J} \mu_j$ be the decomposition of $\tau\,|_{K_1}$ into irreducible pieces. Then canonically $\mathscr{O}(G_1, K_1, V_\tau) = \bigoplus_{j\in J} \mathscr{O}(G_1, K_1, \mu_j)$. Since $\psi_\tau^v(e) = v$ for $v \in V_\tau$, we see that $R_0(H)$ intersects each $\mathscr{O}(G_1, K_1, \mu_j)$ nontrivially, and it then follows that

$$R_0(H) = H(G_1, K_1, V_\tau) = \bigoplus_{j\in J} H(G_1, K_1, V_{\mu_j}). \tag{2.19}$$

The next step is to consider the map R_1 (cf. (1.13)) from H_0 to $\mathscr{O}(G_1, K_1, V_\tau \otimes (p^+/p_1^+)')$. In the present setting, the kernel of R_1 is seen to be

$$H_1 = \{\varphi \in H_0 \mid x \in g^{\mathbf{C}}\colon R_0 r(x)\varphi = 0\}. \tag{2.20}$$

We recall: If a sequence of holomorphic functions $\{f_n\}$ converges to a function f in some region Ω, uniformly on compact sets, then f is holomorphic in Ω, and $\{f_n'\}$ converges to f', uniformly on compact sets (Weierstrass). From this it is easy to see that H_1 is a closed subspace of H_0, that the point-evaluation maps are continuous on $R_1(H_0)$, equipped with the Hilbert space structure from H_0/H_1, and that similar facts are true for all the following steps.

It is then clear that H_n/H_{n+1} can be identified with a subspace of $H(G_1, K_1, V_\tau \otimes (S^{n+1}(p^+/p_1^+))')$. This subspace will in general be proper (see Sections 3 and 4). However, we do have

PROPOSITION 2.5. *Let $\tau \in I \cap P$. Then $H(\tau)$ under the action of G_1 breaks into*

$$H(\tau)\,|_{G_1} = \bigoplus_{n=0}^{\infty} H(G_1, K_1, V_\tau \otimes (S^n(p^+/p_1^+))').$$

Furthermore, the modules on the right-hand side are all finite sums of $H(\mu_i)$'s where $\mu_i \in I_1 \cap P_1 \subset \hat{K}_1$.

Proof. $H_n = \{\varphi \in H(\tau) \mid R_0\varphi = 0$ and $R_0 r(x^\alpha)\varphi = 0$ for all $r(x^\alpha) = r(x_1) \cdots r(x_\alpha)$ with $x_i \in g^{\mathbf{C}}$, $i = 1,..., \alpha, \alpha \leqslant n\}$. We must prove that the map R_{n+1}: $H_n \to H(G_1, K_1, V_\tau \otimes (S^{n+1}(p^+/p_1^+))')$ is surjective. For this, it is sufficient to prove that the map from H_n to $V_\tau \otimes (S^{n+1}(p^+/p_1^+))'$,

$$\varphi \to (R_{n+1}\varphi)(e) \to V_\tau \otimes (S^{n+1}(p^+/p_1^+))',$$

is surjective. That is: for any $v \in V_\tau$ and any set $\{x_1,..., x_{n+1}\} \subset p^+$ there must exist a $\varphi \in H_n$ such that

$$(r(x_1) \cdots r(x_{n+1})\varphi)(e) = v.$$

In the given case we know that the space of K-finite vectors in $H(\tau)$ equals the space of polynomials on D. Hence functions of the form $\varphi(g) = \tau(k(g))^{-1} P(X_+(g))$, where P is any polynomial on D, are all in $H(\tau)$, and we have

$$\begin{aligned}
&(r(x_1) \cdots r(x_n)\varphi)(e) \\
&= \frac{d}{ds_1} \cdots \frac{d}{ds_{n+1}} \varphi(\exp s_1x_1 \cdots \exp s_{n+1}x_{n+1}) \,|_{s_1=\cdots=s_{n+1}=0} \\
&= \frac{d}{ds_1} \cdots \frac{d}{ds_{n+1}} P(s_1x_1 + \cdots + s_{n+1}x_{n+1}) \,|_{s_1=\cdots=s_{n+1}=0}\,.
\end{aligned}$$

This completes the proof since evidently P can be chosen such that the last expression equals v.

COROLLARY 2.6. *Let G be as before, and let $\tau_i \in I \cap P$ for $i = 1, 2$. Then*

$$H(\tau_1) \otimes H(\tau_2) = \bigoplus_{n=0}^{\infty} H(\tau_1 \otimes \tau_2 \otimes S^n(p^-)).$$

Proof. Imbed G in $G \times G$ by the diagonal map and apply Proposition 2.5 with $G \equiv G \times G$ and $G_1 \equiv G$. (Comment: $\tau_1 \otimes \tau_2 \in I \cap P \subset \hat{K}_{G\times G}$.)

We shall now leave the general theory and turn to some special cases particularly important to physics. For these we shall also consider elements τ of $\hat{K}$ that do not belong to P. Typically, in this cas $H(\tau)$ will consist of solutions to certain differential equations, and the restriction to G_1 will then correspond to fixing Cauchy data on D_1. It is then clear that some of the spaces H_n may be rather small, if not zero, and that $H(\tau)$ in some cases will equal the direct sum of only finitely many irreducible representations of G_1.

3. Example: The Mass Zero Equations and $O(2, n+1)$

Consider $\mathbb{R}^{2+(n+1)}$ with basis $(e_{-1}, e_0, e_1, \dots, e_{n+1})$ and let $q(x, x)$ denote the quadratic form which, in the given basis, is given by

$$q(x, x) = x_{-1}^2 + x_0^2 - \sum_{i=1}^{n+1} x_i^2. \tag{3.1}$$

Let $O(2, n+1)$ be the group of linear transformations of $\mathbb{R}^{2+(n+1)}$ leaving q invariant, G_{n+1} its connected component containing the identity, and let $\mathfrak{g}_{n+1}$ denote the Lie algebra of G_{n+1}. The maximal compact subgroup K_{n+1} of G_{n+1} is isomorphic to $SO(2) \times SO(n+1)$, and any irreducible representation τ of K_{n+1} is then of the form $\tau = (\alpha, \mu)$, where the integer α denotes the representation $\alpha\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) = e^{i\alpha\theta}$, and μ is an irreducible representation of $SO(n+1)$ in $V_\tau = V_\mu$. We let z denote the element of $\mathfrak{g}_{n+1}$ which is the generator of the subgroup $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$, and consider, as above, the representation $T_\tau = T_{\alpha,\mu}$ of G_{n+1} in $\mathcal{O}(G_{n+1}, K_{n+1}, V_\tau)$. Let us denote this module by $\mathcal{O}(\alpha, \mu, n+1)$.

The group G_n is imbedded in G_{n+1} by extending its action on $\mathbb{R}^{2+n}$ to $\mathbb{R}^{2+(n+1)}$ in the obvious way. The preceding analysis then shows that the space of K-finite vectors $\mathcal{O}^f(\alpha, \mu, n+1)$ as a $\mathcal{U}(\mathfrak{g}_n)$-module has a filtration for which the set of composition factors is in a one-to-one onto correspondence by isomorphisms to the set

$$\{\mathcal{O}^f(\alpha + i, \mu_r, n) \mid i = 0, 1, 2, \dots \text{ and } r = 1, 2, \dots, l\},$$

where $\mu_1 \oplus \cdots \oplus \mu_l$ is the decomposition of μ's restriction to $SO(n)$ into irreducible representations. As is well known this decomposition is multiplicity free, and it follows that no pair of composition factors are isomorphic as $\mathfrak{g}_n$-modules.

We obtain in particular the following as a corollary.

Corollary 3.1. *Let $\tau \in P$. The restriction of the representation T_τ of G_{n+1} in $H(\tau)$ to G_n is a direct sum of representations, and is multiplicity free.*

We shall analyze the restriction of T_τ to G_n further for particular representations.

Let $M = \mathbb{R}^{1+n}$ denote Minkowski space for $n+1$-dimensional space-time. Let $(e_0, e_1, \dots, e_n)$ be a basis and let

$$x \cdot x = x_0^2 - x_1^2 - \cdots - x_n^2. \tag{3.2}$$

The group G_{n+1} acts on M by (locally defined) conformal transformations. We recall that a space of solutions to the wave equation can be equipped with a

Hilbert space structure in which G_{n+1} acts unitarily [7, 10]. Let us recall the precise construction involved in the above statement.

Consider the quadratic form q' on $\mathbb{R}^{2+(n+1)}$ given by

$$q'(t) = t_0^2 - t_1^2 - \cdots - t_n^2 + t_{-1}t_{n+1}. \tag{3.3}$$

This form has signature $(2, n+1)$ and the linear group that leaves q' invariant is then $O(2, n+1)$. We extend q' to $\mathbb{C}^{2+(n+1)}$ by linearity and consider in the complex projective space the open submanifold

$$D_{n+1} = \{\mathbb{C} \cdot v \mid v \in \mathbb{C}^{2+(n+1)}, q'(v, v) = 0, \text{ and } q'(v, \bar{v}) > 0\}. \tag{3.4}$$

If $v = z_{-1}e_{-1} + z_0e_0 + \cdots + z_{n+1}e_{n+1}$ then $\mathbb{C} \cdot v$ is in D_{n+1} if and only if

$$z_{-1}z_{n+1} + z \cdot z = 0, \tag{3.5a}$$

$$\tfrac{1}{2}(z_{-1}\bar{z}_{n+1} + \bar{z}_{-1}z_{n+1}) + |z_0|^2 - |z_1|^2 - \cdots - |z_n|^2 > 0 \tag{3.5b}$$

where $z \cdot z = z_0^2 - z_1^2 - \cdots - z_n^2 = z^2$.

These equations imply that z_{-1} (as well as z_{n+1}) is different from zero, and we can then normalize v such that $z_{-1} = 1$. If v is normalized we let

$$T(\mathbb{C} \cdot v) = T(v) = (z_0, z_1, \ldots, z_n) \tag{3.6}$$

and can then parametrize D by the subset $\Omega_{n+1} = T(D_{n+1}) \subseteq \mathbb{C}^{n+1}$. Expressed on Ω_{n+1} Eq. (3.5b) becomes

$$(\operatorname{Im} z_0)^2 > (\operatorname{Im} z_1)^2 + \cdots + (\operatorname{Im} z_n)^2. \tag{3.7}$$

Ω_{n+1} is a domain with two connected components Ω_{n+1}^+ and Ω_{n+1}^- (corresponding to $\operatorname{Im} z_0 \lessgtr 0$). We consider the domain Ω_{n+1}^+. Its Shilov boundary is the set $\{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \operatorname{Im} z_i = 0,\ \ i = 0, 1, \ldots, n\} = \mathbb{R}^{n+1} = M$. Finally, if $(z_0, z_1, \ldots, z_n)$ is in Ω_{n+1}, we let

$$v(z_0, z_1, \ldots, z_n) = e_{-1} + z_0e_0 + \cdots + z_ne_n - (z \cdot z)\, e_{n+1}. \tag{3.8}$$

The group G_{n+1} acts on $\mathbb{C}^{2+(n+1)}$ by $v \to g \cdot v$. If $v \to \tilde{v}$ denotes the natural map from $\mathbb{C}^{2+(n+1)}$ onto the projective space $\mathbb{P}^{2+(n+1)}$ then $g \cdot \tilde{v} = \widetilde{g \cdot v}$ defines an action of G_{n+1} on this space which clearly preserves D_{n+1}, and which in fact, since G_{n+1} is connected, leaves both connected components, D_{n+1}^+ and D_{n+1}^-, of D_{n+1} invariant.

We now define a function j on $\mathbb{C}^{2+(n+1)}$ by, if $v = z_{-1}e_{-1} + \cdots + z_{n+1}e_n$,

$$j(v) = z_{-1}. \tag{3.9}$$

The function

$$j(g, \tilde{v}) = j(g \cdot v)\, j(v)^{-1} \tag{3.10}$$

is then well defined on $G_{n+1} \times D_{n+1}$ and there satisfies

$$j(g_1 g_2, \tilde{v}) = j(g_1, g_2\tilde{v})\, j(g_2, \tilde{v}). \tag{3.11}$$

This means that we can define a family of representations T_α of G_{n+1} on $\mathcal{O}(D_{n+1})$ (or $\mathcal{O}(D^+_{n+1})$) by

$$(T_\alpha(g) f)(\tilde{v}) = j(g^{-1}, \tilde{v})^{-\alpha} f(g^{-1}\tilde{v}) \tag{3.12}$$

for $\alpha \in \mathbb{Z}$.

As in Section 2, we denote by $\mathcal{O}^f(\alpha; n + 1)$ the space of K-finite vectors of the representation T_α of the group G_{n+1} acting on $\mathcal{O}(D^+_{n+1})$, and by $W(\alpha; n + 1)$ the minimal submodule of $\mathcal{O}^f(\alpha; n + 1)$. We shall below realize these last modules as modules of functions on Ω^+_{n+1}, the unbounded realization of G/K.

We consider the function R on $\Omega_{n+1} \times \bar{\Omega}_{n+1}$,

$$R(z, z') = q'(v(z), \overline{v(z')}) = -\tfrac{1}{2}(z - \bar{z}')^2, \tag{3.13}$$

and define

$$j(g, z) = j(g, \widetilde{v(z)}). \tag{3.14}$$

Since by definition $\widetilde{g \cdot v(z)} = \widetilde{v(gz)}$, $g \cdot v(z) = j(g \cdot v(z))\, v(gz)$. Hence

$$R(gz, gz') = j(g, z)^{-1} R(z, z')\, \overline{j(g, z')}^{-1}. \tag{3.15}$$

The stabilizer K of $i = (i, 0, 0,\ldots, 0) \in \Omega^+$ in G_{n+1} is isomorphic to $SO(2) \times SO(n + 1)$. Hence under T_α's restriction to K the function $z \to K(z, i)^{-\alpha}$ on Ω^+_{n+1} transforms by a character of K, and we obtain the irreducible module $W(\alpha, n + 1)$ by letting $\mathcal{U}(g_{n+1})$ act on it.

If $\alpha \geqslant (n - 1)/2$, it is known that $\alpha \in P$ and if $\alpha > (n - 1)/2$, $\alpha \in I \cap P$. We denote in this case the representation of G_{n+1} in the space $H(\alpha; n + 1)$ of holomorphic functions on Ω^+_{n+1} by $T(\alpha; n + 1)$.

We are interested in a study of the value $\alpha_0 = (n - 1)/2$; in this case we have the following inclusion: $W(\alpha_0; n + 1) \subsetneqq \mathcal{O}^f(\alpha_0; n + 1)$. We will study the decomposition under g_n of both of these modules.

Let us consider the wave operator

$$\Box = \frac{\partial^2}{\partial z_0{}^2} - \frac{\partial^2}{\partial z_1{}^2} - \cdots - \frac{\partial^2}{\partial z_n{}^2} \tag{3.16}$$

on Ω_{n+1}. Since (with z^2 as in (3.5))

$$\Box(z^2)^{-\alpha} = -2\alpha(n-1-2\alpha)(z^2)^{-\alpha-1}; \tag{3.17}$$

$(z^2)^{-\alpha}$, for $\alpha = (n-1)/2$, is a solution to the wave equation. Moreover, the module spanned by the function $k(z, i)^{-((n-1)/2)} = (-\underline{h}(z+i)^2)^{-((n-1)/2)}$ under the action of $\mathscr{U}(g_{n+1})$ consists of all the k-finite functions φ on $\mathscr{O}(\Omega_{n+1}^+)$ that are solutions to $\Box\varphi = 0$. We let $H(\frac{n-1}{2})$ denote the corresponding Hilbert space of holomorphic functions on Ω_{n+1}^+, and let $U_{(n-1)/2}$ denote the representation of G_{n+1} in $H(\frac{n-1}{2})$.

PROPOSITION 3.2. *The retriction of $U_{(n-1)/2}$ to G_n is the direct sum of the representations $T(\frac{n-1}{2}, n)$ and $T(\frac{n+1}{2}, n)$.*

Proof. Consider the filtration of $H(\frac{n-1}{2})$ by the closed subspaces

$$H\left(\frac{n-1}{2}\right)_r = \left\{\varphi \in H\left(\frac{n-1}{2}\right) \,\Big|\, \left(\frac{\partial}{\partial z_n}\right)\beta_\varphi \Big|_{z_n=0} = 0 \text{ for } 0 \leqslant \beta \leqslant r\right\}.$$

These spaces are invariant under G_n, and clearly, for $r \geqslant 1$, $H(\frac{n-1}{2})_r$ is zero since if φ, as well as $\partial\varphi/\partial z_n$, is zero on the subset $z_n = 0$ and if $\Box\varphi = 0$, then φ is zero everywhere. It thus only remains to be seen that there exists a K-finite function in $H(\frac{n-1}{2})$ which vanishes on the set $z_n = 0$, but this follows since the operator $\partial/\partial z_n$ on holomorphic functions equals $\partial/\partial x_n$ where $z_n = x_n + iy_n$, and the latter comes from the subgroup of translations on Minkowski space. Thus $(\partial/\partial z_n)((z+i)^2)^{-((n-1)/2)}$ belongs to $H(\frac{n-1}{2})$ and is clearly proportional to $z_n((z+i)^2)^{-((n+1)/2)}$.

The space of restrictions to Minkowski space of the span of all K-finite vectors of the representations T_α, $\alpha \geqslant \frac{n-1}{2}$, is a space of analytic functions. We denote the restriction map by RM. The subspace $RM(\mathscr{O}^f(\frac{n-1}{2}; n+1)$ is the positive energy subspace consider in [6] for $n = 3$.

The preceding analysis leads to

PROPOSITION 3.3. *$\mathscr{O}^f(\frac{n-1}{2}, n+1)$ on Ω_{n+1}^+ splits under $\mathscr{U}(g_n)$ into a direct sum of the modules $W(\frac{n-1}{2}+\lambda, n)$, $\lambda = 0, 1, 2, \ldots$. Each of these submodules is unitarizable as a g_n-module. The highest weight vector in $W(\frac{n-1}{2}+\lambda, n)$ under the action of g_n is the function $z_n{}^\lambda((z+i)^2)^{-(\lambda+(n-1)/2)}$.*

Proof. We have seen that $\mathscr{O}^f(\frac{n-1}{2}, n+1)$ on Ω_{n+1}^+ has a composition series given by the modules $\mathscr{O}^f(\frac{n-1}{2}+\lambda, n)$ with respect to g_n. But if $\alpha = \frac{n-1}{2}+\lambda$ with

$\lambda \geqslant 0$, then clearly $\alpha > \frac{(n-1)-1}{2}$, where $\frac{(n-1)-1}{2}$ is the parameter for the wave equation for G_n. Hence $\mathcal{O}^f(\frac{n-1}{2} + \lambda, n) = W(\frac{n-1}{2} + \lambda, n)$, and the conclusion now follows from Corollary 2.4.

COROLLARY 3.4. *Let $\varphi \in RM(\mathcal{O}^f(\frac{n-1}{2}, n+1))$. Then, considered as a distribution, the Fourier transform φ is supported by the cone $C^+ = \{k \in \mathbb{R}^{n+1}; k_0^2 \geqslant k_1^2 + \cdots + k_n^2; k \geqslant 0\}$.*

Proof. The decomposition of $\mathcal{O}^f(\frac{n-1}{2}, n+1)$ under g_n is $\bigoplus_\lambda W(\frac{n-1}{2} + \lambda, n)$. We have $W(\frac{n-1}{2} + \lambda, n) = \mathcal{U}(g_n) \cdot (x_n{}^\lambda((x+i)^2)^{-\lambda+(n-1)/2}$. Compute the Fourier transform of the function $x_n{}^\lambda((x+i)^2)^{-\lambda+(n-1)/2}$. If $z \in \Omega_{n+1}^+$, $(z^2) \neq 0$ and $(z^2)^{-s+(n-1)/2} = m(s)\int_{C^+} e^{i\langle k,z\rangle}(k^2)^{s-1}\,dk$ whenever $s > 0$. The function $m(s)$ has a simple zero for $s = 0$.

Let us consider $b(C^+) = \{k; k_0^2 = k_1^2 + \cdots + k_n^2; k_0 \geqslant 0\}$, the boundary of the cone C^+, equipped with the surface measure $dm(k)$. If we consider the distribution f^s defined by the function $(k^2)^{(s-1)}$ restricted to C^+, for $s > 0$, this distribution has a meromorphic continuation in s and $s\langle\varphi, f^s\rangle$ tends to $\int_{b(C^+)} \varphi(k)\,dk(k)$ when $s \to 0$. From

$$\left(\frac{\partial}{\partial k_m}\right)^p (k^2)^{s-1} = (-2)^p(s-1)\cdots(s-p)\,k_n^p(k^2)^{s-1-p} + \sum_{i<p} p_i(\ ,\ )(k^2)^{s-1-i}$$

where $p_i(s, k_n)$ is a polynomial in s and k_n of degree i, we see that for $p < s$, the Fourier transform of $x_n^p((x+i)^2)^{-(s+(n-1)/2)}$ is supported in the interior of the cone C^+. Letting $s \to p$ we see that the Fourier transform of our function is a sum of a function supported by $b(C^+)$ and a function supported in the interior of C^+.

Now consider the subgroup P'_{n+1} of $O(2, n+1)$ of affine transformations of the domain Ω_{n+1}^+, and denote by p'_{n+1} its Lie algebra. We have $g_n = p'_n + k_n$ by the Iwasawa decomposition, hence since the function $x_n{}^\lambda((x+u)^2)^{-\lambda+(n-1)/2}$ transforms by a character of $k_n \simeq SO(2)\times SO(n)$, we have $W(\lambda + \frac{n-1}{2}, n) = \mathcal{U}(p'_n) \cdot (x_n{}^\lambda(x+i)^2)^{-\lambda+(n-1)/2}$.

As the action of $\mathcal{U}(p'_n)$ is given by differential operators with polynomial coefficients, the corollary follows.

4. EXAMPLE: TENSOR PRODUCTS OF ANALYTIC CONTINUATIONS OF THE HOLOMORPHIC DISCRETE SERIES FOR $SU(2, 2)$

In this example we analyze tensor products of highly singular type for $SU(2, 2)$. Some of the material applies equally well to $SU(n, n)$ and $SP(n, \mathbb{R})$, and indicates a direction of attack which may work for other singular restriction problems. In the end, however, it turns out that to get even some of the simplest cases, a detailed knowledge of the representations of holomorphic type of $SU(2, 2)$ and some concrete computations are needed.

For this case it is easiest to work with the unbounded realization of the associated Hermitian symmetric space. Specifically, we take the domain to be $D \times D$, where $D = \{z \in gl(2, \mathbb{C}) \mid (z - z^*)/2i > 0\}$, and we restrict to the diagonal in $D \times D$, which we identify with D. We recall from [5] that if $(z_1, z_2) \in D \times D$, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SU(2, 2)$, and if $y = (z_1 - z_2)/2$, then for $g \cdot z = (az + b)(cz + d)^{-1}$

$$\left(\frac{gz_1 - gz_2}{2}\right) = (z_2 c^* + d^*)^{-1} y (cz_1 + d)^{-1}. \tag{4.1}$$

From now on we let $G = SU(2, 2)$, and let K be the maximal compact subgroup.

To begin with, we need some observations:

If τ is a finite-dimensional unitary representation of K in V_τ, we identify $\mathcal{O}(G, K, V_\tau)$ with the space $\mathcal{O}(D, V_\tau)$ of holomorphic functions from D to V_τ and let, as before, T_τ denote the corresponding action of G on $\mathcal{O}(D, V_\tau)$.

Let us be explicit in this case: We have $K = \{\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) \mid ((a + ib), (a - ib)) \in U(2) \times U(2)$ and $\det(a + ib)(a - ib) = 1\}$. Let $u_1 = a + ib$, and $u_2 = a - ib$. Then τ is of the form

$$\tau = \bigoplus_{i=1}^{n} \left(\det u_1^{\delta_i} \left(\bigotimes_s^{\alpha_i} u_1\right) \otimes \left(\bigotimes_s^{\beta_i} u_2\right)\right) \tag{4.3}$$

where $\otimes_s^{\alpha_i}$ denotes the α_ith fold symmetrized tensor product. Then with

$$J_\tau(g, z) = \bigoplus_{i=1}^{n} \left(\det(cz + d)^{\delta_i} \left(\bigotimes_s^{\alpha_i} (cz + d)\right) \otimes \left(\bigotimes_s^{\beta_i} (zc^* + d^*)^{-1}\right)\right), \tag{4.4}$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, the representation T_τ on $\mathcal{O}(D, V_\tau)$ is given by

$$(T_\tau(g) f)(z) = J_\tau(g^{-1}, z)^{-1} f(g^{-1} z). \tag{4.5}$$

LEMMA 4.1. *Let τ_1 and τ_2 be given finite-dimensional unitary representations of K, and let $\tau_1 \otimes \tau_2 = \otimes_{j \in J} \tau_j$ be the decomposition of $\tau_1 \otimes \tau_2$, as a representation of K, into irreducible subrepresentations. Suppose that there exists a reproducing kernel Hilbert space H of holomorphic functions from D to $V_{\tau_1} \otimes V_{\tau_2}$ such that $T_{\tau_1 \otimes \tau_2}$ is unitary in H, and let $H = \bigoplus_{j \in \tau} H_j$ be the decomposition of H corresponding to $T_{\tau_1 \otimes \tau_2} = \bigoplus_{j \in J} T_{\tau_j}$. Then the Hilbert space structures on the H_j's can be scaled in such a way that the reproducing kernel $K\colon D \times D \to \mathrm{Aut}(V_{\tau_1} \otimes V_{\tau_2})$ is the restriction of a function $K_\otimes\colon D \times D \times D \to \mathrm{Aut}(V_{\tau_1} \otimes V_{\tau_2})$, holomorphic in the two first variables, antiholomorphic in the last, in the sense that*

$$\forall v \in V_{\tau_1} \otimes V_{\tau_2}, \qquad \forall w \in D\colon K(z, w)v = (R_0 K_\otimes(\cdot, \cdot, w)v)(z) \tag{4.6}$$

and such that moreover

$$\forall v \in V_{\tau_1} \otimes V_{\tau_2}, \qquad \forall w \in D:$$

$$((T_{\tau_1} \otimes T_{\tau_2})(g)\, K_{\otimes}(\cdot, \cdot, w)v)(z_1, z_2)$$

$$= K_{\otimes}(z_1, z_2, gw)\, J_{\tau_1 \otimes \tau_2}(g, w)^{-1*}v. \tag{4.7}$$

Proof. Let K be a reproducing kernel on H. Then we have [5], for $g \in G$, and all z, w in D

$$K(gz, gw) = J_{\tau_1 \otimes \tau_2}(g, z)\, K(z, w)\, J_{\tau_1 \otimes \tau_2}(g, w)^*. \tag{4.8}$$

In particular, for all k in K

$$K(i, i) = ((\tau_1 \otimes \tau_2)(k))\, K(i, i)((\tau_1 \otimes \tau_2)(k))^{-1}. \tag{4.9}$$

$K(i, i)$ is a positive (self-adjoint) operator, and we see from (4.9) that each eigenspace of $K(i, i)$ is invariant under $\tau_1 \otimes \tau_2$. Hence $K(i, i)$ commutes with the extension of $\tau_1 \otimes \tau_2$ to $GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$ (cf. (4.3)). $K(z, w)$ is completely determined by $k(z, z)$, and it follows from (4.8), (4.4), and the above that

$$k(z, z) = J_{\tau_1 \otimes \tau_2}(g, i)\, J_{\tau_1 \otimes \tau_2}(g, i)^* k(i, i) \tag{4.10}$$

where $g \cdot i = z$. It is now clear that $k(i, i)$ can be chosen to be the identity operator on $V_{\tau_1} \otimes V_{\tau_2}$. Thus we shall assume that

$$k(z, z) = J_{\tau_1 \otimes \tau_2}(g, i)\, J_{\tau_1 \otimes \tau_2}(g, i)^*, \tag{4.11}$$

where $g \cdot i = z$. Since $J_{\tau_1 \otimes \tau_2}(g, z) = J_{\tau_1}(g, z) \otimes J_{\tau_2}(g, z)$ we can now define a function $K'_{\otimes}(z_1, z_2, w_1, w_2)$ on $D \times D \times D \times D$; holomorphic in z_1, z_2, and antiholomorphic in w_1, w_2 by its value on the diagonal

$$K'_{\otimes}(z_1, z_2, z_1, z_2) = J_{\tau_1}(g_1, i)\, J_{\tau_1}(g_1, i)^* \otimes J_{\tau_2}(g_2, i)\, J_{\tau_2}(g_2, i)^* \tag{4.12}$$

where $g_1 \cdot \mathrm{i} = z_1$ and $g_2 \cdot i = z_2$. This is clearly well defined and $K_{\otimes}(z_1, z_2, w) = K'_{\otimes}(z_1, z_2, w, w)$ is then the desired function.

Finally, the following will be useful:

LEMMA 4.2. *Let T denote a representation of G in the space of holomorphic functions from $D \times D$ to some finite-dimensional vector space V and assume that there are reproducing kernel Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, both contained in $\mathfrak{A}_{n-1}$* (1.14), *to which the restriction of T is unitary and irreducible. Assume moreover that there exists an irreducible unitary representation T_n of G in a reproducing* *kernel Hilbert space* $\mathscr{H}_n$ *of holomorphic functions from* D *to* V *such that* $R_n(\mathscr{H}_1) = R_n(\mathscr{H}_2) = \mathscr{H}_n$ (cf. 1.15), *and*

$$R_n(T\,|_{\mathscr{H}_1}) = R_n(T\,|_{\mathscr{H}_2}) = T_n R_n\,.$$

Then $\mathscr{H}_1 = \mathscr{H}_2$.

Remark. This means that they coincide as sets and that the Hilbert space structures are proportional.

Proof. If $\mathscr{H}_1 \cap \mathscr{H}_2 \neq \{0\}$ it is obvious by irreducibility, so we assume that $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$. Then the map S from $\mathscr{H}_1 \oplus \mathscr{H}_2$ to the space of holomorphic functions on $D \times D$ with values in V given by

$$S(f, g) = f + g$$

is injective. The range $S(\mathscr{H}_1 \oplus \mathscr{H}_2)$ can therefore be equipped with a Hilbert space structure. We let $\mathscr{H}$ denote this space. It is easy to see that point evaluation is continuous and hence that $\mathscr{H}$ is a reproducing kernel Hilbert space which accordingly can be decomposed by differentiation and restriction. We know that we must pick up $T_n \oplus T_n$ by this procedure, but in the step where we use R_n (which is the first nontrivial step) we clearly only get T_n. This means that the other T_n must be picked up in a later step, which, however, is impossible, since the K-types change under differentiation.

The following is then obvious:

COROLLARY 4.3. *In Lemma* 4.2 *the group* G *can be replaced by* K *provided one assumes that* $\dim \mathscr{H}_1 = \dim \mathscr{H}_2 < \infty$. (*And maintain the other hypotheses.*)

Remark. This does not imply that there are no multiplicities in the K-types.

We are now ready to turn to the topic of this section: Let T_{τ_1} and T_{τ_2} be two irreducible unitary representations of G in reproducing kernel Hilbert spaces $H(\tau_1)$ and $H(\tau_2)$ consisting of holomorphic functions from D to V_{τ_1} and V_{τ_2}, respectively, and consider $T_{\tau_1} \otimes T_{\tau_2}$ acting unitarily in the reproducing kernel Hilbert space $H(\tau_1) \otimes H(\tau_2)$ of holomorphic functions from $D \times D$ to $V_{\tau_1} \otimes V_{\tau_2}$.

As in [5] we introduce variables z and y on $D \times D$ where, for $(z_1, z_2) \in D \times D$

$$z = \frac{z_1 + z_2}{2} \qquad \text{and} \qquad y = \frac{z_1 - z_2}{2}. \tag{4.13}$$

We let

$$H_r = \Big\{\varphi \in H_{\tau_1} \otimes H_{\tau_2} \mid R_0\varphi = 0 \text{ and } R_0\left(\frac{\partial^{\alpha_1}}{\partial y_1} \cdots \frac{\partial^{\alpha_4}}{\partial y_4}\right)\varphi = 0$$
$$\text{for all } \alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leqslant r\Big\}, \tag{4.14}$$

and define maps R_r analogous to the preceding cases. If is also convenient to introduce the notation Q_{r+1} for the orthogonal complement of H_{r+1} in H_r. Finally, let $\mathcal{O}(D \times D, V_{\tau_1} \otimes V_{\tau_2})$ denote the module of all holomorphic functions from $D \times D$ to $V_{\tau_1} \otimes V_{\tau_2}$, let $\mathcal{O}_{r+1}(D \times D, V_{\tau_1} \otimes V_{\tau_2})$ be the analog of H_r, and let $\mathcal{O}^f(D \times D, V_{\tau_1} \otimes V_{\tau_2})$ denote the space of K-finite vectors in $\mathcal{O}(D \times D, V_{\tau_1} \otimes V_{\tau_2})$.

Let $N(r)$ denote the dimension of the space of homogeneous polynomials in four variables of degree r ($N(r) = \binom{r+3}{3}$). We identify this space with $\otimes_s^r gl(2, \mathbb{C})$ and let $T^r(\tau_1, \tau_2)$ denote the action of G on the space of functions from $D \times D$ to $\otimes_s^r gl(2, \mathbb{C}) \otimes V_{\tau_1} \otimes V_{\tau_2}$, which on functions of the form $f_M \otimes h$, where $f_M(z_1, z_2) = M(z_1, z_2) \otimes \cdots \otimes M(z_1, z_2)$ is a function from $D \times D$ to $\otimes_s^r gl(2, \mathbb{C})$ and h is a function from $D \times D$ to $V_{\tau_1} \otimes V_{\tau_2}$, is given by

$$\begin{aligned}(T^r(\tau_1, \tau_2)(g) f_M \otimes h)(z_1, z_2) &= \bigotimes_s^r ((cz_1 + d)^{-1} M(g^{-1}z_1, g^{-1}z_2)(z_2 c^* + d^*)^{-1}) \\ &\quad \otimes ((T_{\tau_1} \otimes T_{\tau_2})(g)h)(z_1, z_2) \qquad (4.15)\end{aligned}$$

for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

Let us assume that there is a linear subspace $S_0{}^r$ of $\mathcal{O}(D \times D, \otimes_s^r gl(2, \mathbb{C}) \otimes V_{\tau_1} \otimes V_{\tau_2})$ such that:

$$\text{If } f \in S_0{}^r \text{ and } R_0 f = 0 \text{ then } f = 0. \qquad (4.16a)$$

$$S_0{}^r \text{ is invariant under } T^r(\tau_1, \tau_2)(g) \text{ for all } g \text{ in } G. \qquad (4.16b)$$

As a basis of $\otimes_s^r gl(2, \mathbb{C})$ we choose

$$e_i = e_{i_1, i_2, i_3, i_4} = P_s(\underbrace{M_1 \otimes \cdots \otimes M_1}_{i_1} \otimes \cdots \otimes \underbrace{M_4 \otimes \cdots \otimes M_4}_{i_4}) \qquad (4.17)$$

where $M_1, \ldots, M_4$ is a basis of $gl(2, \mathbb{C})$, $i_1 + i_2 + i_3 + i_4 = r$, and P_s denotes the symmetrization map (projection).

Any function F in $\mathcal{O}(D \times D, \otimes_s^r gl(2, \mathbb{C}) \otimes (V_{\tau_1} \otimes V_{\tau_2}))$ can uniquely be written as

$$F(z_1, z_2) = \sum_{i=1}^{N(r)} e_i \otimes f_i(z_1, z_2) \qquad (4.18)$$

where f_i is a holomorphic function for $i = 1, 2, \ldots, N(r)$. Hence we can define a map $P^r(\tau_1, \tau_2)$ from $S_0{}^r$ to $\mathcal{O}_r(D \times D, V_{\tau_1} \otimes V_{\tau_2})$ by

$$\begin{aligned}(P^r(\tau_1, \tau_2) F(z_1, z_2) \\ = \sum_{i=1}^{N(r)} (\operatorname{tr} yM_1)^{i_1} (\operatorname{tr} yM_2)^{i_2} (\operatorname{tr} yM_3)^{i_3} (\operatorname{tr} yM_4)^{i_4} f_i(z_1, z_2). \qquad (4.19)\end{aligned}$$

LEMMA 4.4. (a) $P^r(\tau_1, \tau_2)\, T^r(\tau_1, \tau_2) = T_{\tau_1} \otimes T_{\tau_2} P^r(\tau_1, \tau_2)$

(b) *$P^r(\tau_1, \tau_2)$ is injective.*

Proof. (a) is essentially contained in (4.1).

To prove (b) assume that $P^r(\tau_1, \tau_2)F = 0$. Then clearly $R_r P^r(\tau_1, \tau_2)F = 0$ and since this equals $R_0 F$, F is zero by assumption.

One application of this lemma is the following: Consider $T^r(\tau_1, \tau_2)$ and let $\tau_1{}^r$ and $\tau_2{}^r$ be the representations of K given by

$$\tau_1{}^r(u_1, u_2) = \underbrace{(u_1 \otimes \cdots \otimes u_1)}_{r} \otimes \tau_1(u_1, u_2),$$

and (4.20)

$$\tau_2{}^r(u_1, u_2) = (\det u_2)^{-r} \underbrace{(u_2 \otimes \cdots \otimes u_2)}_{r} \otimes \tau_2(u_1, u_2).$$

Then $T^r(\tau_1, \tau_2)$ is in a natural way contained in $T_{\tau_1{}^r} \otimes T_{\tau_2{}^r}$. Of course, $\tau_1{}^r$ and $\tau_2{}^r$ are not irreducible representations, but writing them as finite sums of elements of $\hat{K}$, we see that $T^r(\tau_1, \tau_2)$ is contained in a direct sum of representations $T_{\mu_1{}^r} \otimes T_{\mu_2{}^r}$ where $\mu_1{}^r$, as well as $\mu_2{}^r$, is contained in $\hat{K}$.

COROLLARY 4.5. *Let $T_{\mu_1{}^r}$ and $T_{\mu_2{}^r}$ be as above. If there are reproducing kernel Hilbert spaces $H(\mu_1{}^r)$ and $H(\mu_2{}^r)$ of holomorphic functions in which $T_{\mu_1{}^r}$ and $T_{\mu_1{}^r}$, respectively, are unitary, and if $T_{\mu_1{}^r} \otimes T_{\mu_1{}^r}$ is contained in $T^r(\tau_1, \tau_2)$, then $T_{\tau_1} \otimes T_{\tau_2}$ is unitary in a reproducing kernel Hilbert space of holomorphic functions. Restricted to this space, it is unitarily equivalent to $T_{\mu_1{}^r \otimes \mu_2{}^r}$.*

Proof. The orthogonal complement in $H(\mu_1{}^r) \otimes H(\mu_2{}^r)$ of the space of functions that vanish on the diagonal is a reproducing kernel Hilbert space. If we apply $P^r(\tau_1, \tau_2)$ the resulting space is one in which point-evaluation is continuous.

If we only demand that the space $S_0{}^r$ (4.16) should be invariant under K then $P^r(\tau_1, \tau_2)$ can be used, by means of Lemma 4.1 and Corollary 4.3, to compare minimal K-types. This is used in an example below.

We shall say that a representation $T(\tau)$ in $\mathcal{O}(D, V_\tau)$ is supported by the forward light cone $C^+ = \{k \in H(2) \mid \det k > 0,\ \operatorname{tr} k > 0\}$ if it is unitary in a space $H(\tau) \subset \mathcal{O}(D, V_\tau)$ and if the elements of $H(\tau)$ are Fourier–Laplace transforms of functions from C^+ to V_τ.

If the above space $H(\tau)$ is a reproducing kernel Hilbert space, it is easy to see that there exists a continuous function $F_\tau\colon C^+ \to \operatorname{Aut}(V_\tau)$ such that the kernel K_τ is given by

$$K_\tau(z, w) = \int_{C^+} F_\tau(k)\, e^{i\operatorname{tr}(z-w^*)k}\, dk. \tag{4.21}$$

If, for all $k \in C^+$, $F_\tau(k)$ is nonsingular we say that $T(\tau)$ is strongly supported by C^+. By restricting to the point $i \in D$ the K-types in $H(\tau)$ of a representation $T(\tau)$ which is strongly supported by C^+ can be found by arguments analogous to those in the proof of theorem 3.2 in [5].

PROPOSITION 4.6. *If $\tau \in K$ and if $T(\tau)$ is strongly supported by C^+, then $\mathcal{O}^f(D, V_\tau) \subseteq H(\tau)$.*

We shall now turn to some concrete example of the above. We recall from [1, 2, 7] that if τ denotes the defining representation of $GL(2, \mathbb{C})$; $\tau(g) = g$, if τ_n denotes the nth fold symmetrized tensor product of τ, and if $\tau_0 = 1$, then one can define two series $T_1(n, \alpha)$ and $T_2(m, \beta)$ of representations of G by

$$(T_1(n, \alpha)(g) f)(z) = \tau_n(cz + d)^{-1} \det(cz + d)^{-(\alpha+2)} f(g^{-1}z), \tag{4.22}$$

and

$$(T_2(m, \beta)(g) f)(z) = \tau_m(zc^* + d^*) \det(zc^* + d^*)^{-(\beta+m+2)} f(g^{-1}z),$$

for $\alpha, \beta \in \mathbb{Z}$, and $n, m \geqslant 0$. For $\alpha > -1$ and $\beta > -1$ the representations $T_1(n, \alpha)$ and $T_2(m, \beta)$ are unitary, irreducible, and strongly supported by C^+ in reproducing kernel Hilbert spaces $H_1(n, \alpha)$ and $H_2(m, \beta)$, respectively. $T_1(n, -1)$ and $T_2(m, -1)$ are unitary and irreducible in reproducing kernel Hilbert spaces $H_1(n, -1)$ and $H_2(m, -1)$, respectively, where $H_1(n, -1)$, as well as $H_2(m, -1)$, consists of holomorphic solutions to certain wave equations.

Finally, we define

$$\begin{aligned}(T_3(n, m, \gamma)(g) f)(z) \\ = \tau_n(cz + d)^{-1} \otimes \tau_m(zc^* + d^*) \det(cz + d)^{-\gamma} f(g^{-1}z).\end{aligned} \tag{4.23}$$

PROPOSITION 4.7 [2]. *For $\gamma \geqslant m + 2$, $T_3(n, m, \gamma)$ is unitary and irreducible in a reproducing kernel Hilbert space $H_3(n, m, \gamma)$.*

Proof. For such γ the representation $T_3(n, m, \gamma)$ can be obtained as the restriction to the diagonal of $T_1(n, \alpha) \otimes T_2(m, \beta)$ $\alpha, \beta \geqslant -1$, since $\det(zc^* + d^*) = \det(cz + d)$.

LEMMA 4.8 [2]. *For $\gamma \geqslant m + 2$, $T_3(n, m, \gamma)$ is supported by C^+.*

Proof. The reproducing kernel is given by

$$\tau_n \left(\frac{z - w^*}{2i}\right)^{-1} \otimes \tau_m \left(\frac{z - w^*}{2i}\right) \det \left(\frac{z - w^*}{2i}\right)^{-\gamma} \tag{4.24}$$

Let $\epsilon = \gamma - m - 2$. Then the kernel is given by $M_{n,m}(\frac{z-w^*}{2i}) \det(\frac{z-w^*}{2i})^{-2-\epsilon-m-n}$ where $M_{n,m}(\frac{z-w^*}{2i})$ is a matrix whose entries are polynomials of degree $n + m$ in the entries of $(\frac{z-w^*}{2i})$. We know [7] that there are constants c_ρ such that for $\rho \geqslant 0$

$$\det\left(\frac{z-w^*}{2i}\right)^{-\rho-2} = c_\rho \int_{C^+} e^{i\operatorname{tr}(z-w^*)k} \det k^\rho \, dk \tag{4.25}$$

and the proof is completed by observing that the kernel can be obtained by differentiation of $\det(\frac{z-w^*}{2i})^{-2-\epsilon}$ modulo some correction terms obtainable by differentiation of functions of the form $\det(\frac{z-w^*}{2i})^{-2-\epsilon-i}$ for $i \geqslant 1$.

Remark. It is proved in [2] that for $\gamma > m+2$, $T_3(n, m, \gamma)$ is strongly supported by C^+, whereas for $\gamma = m+2$ it is not.

Let T_{τ_1} and T_{τ_2} be two limit points of the series (4.22). It then remains to be seen how $H(\tau_1) \otimes H(\tau_2)$ decomposes. We give a typical example of how the above approach leads to the answer:

EXAMPLE 4.1. $T_1(n, -1) \otimes T_1(m, -1)$. The reproducing kernel K_p' for $T_1(p, -1)$ is given by [7, p. 75]

$$K_p'(z, w) = \int_{s(C^+)} \tau_n(k)\, e^{i\operatorname{tr}(z-w^*)k}\, dm(k). \tag{4.26}$$

Consider $T^r(\tau_1, \tau_2)$ and let $T_0{}^r(\tau_1, \tau_2)$ be the representation defined by $R_0 T^r(\tau_1, \tau_2) = T_0{}^r(\tau_1, \tau_2) R_0$. It is straightforward to see that the corresponding kernel $K_{\otimes}^r(z_1, z_2, w)$ (Lemma 4.1) is the operator which maps $(\otimes_s^r M) \otimes v_1 \otimes v_2$ into

$$\bigotimes_s^r \left(\left(\frac{z_1 - w^*}{2i}\right)^{-1} M \left(\frac{z_2 - w^*}{2i}\right)^{-1}\right) \otimes K_n'(z_1, w)\, v_1 \otimes K_m'(z_2, w)\, v_2 \tag{4.27}$$

for $M \in gl(2, \mathbb{C})$, $v_1 \in \otimes_s^n \mathbb{C}^2$, and $v_2 \in \otimes_s^m \mathbb{C}^2$. The map P^r can be chosen in such a way that the expression in (4.27) is mapped into

$$\left(\operatorname{tr}\left(\frac{z_1 - z_2}{2}\right)\left(\frac{z_1 - w^*}{2i}\right)^{-1} M \left(\frac{z_2 - w^*}{2i}\right)^{-1}\right) \otimes K_n'(z_1, w) v_1 \otimes K_m'(z_2, w) v_2 \tag{4.28}$$

and the question becomes: For which linear combinations of elements of $\otimes_s^r gl(2, \mathbb{C}) \otimes (\otimes_s^n \mathbb{C}^2) \otimes (\otimes_s^m \mathbb{C}^2)$ do the corresponding linear combinations of the functions (4.28) belong to $H_1(n, -1) \otimes H_1(m, -1)_*$ This may seem, especially for large n and m's, to be a hopeless case. However, there are some obvious combinations that do work: We write $(z_1 - z_2) = (z_1 - w^*) - (z_2 - w^*)$ and obtain:

$$\begin{aligned} &\operatorname{tr}((z_1 - z_2)(z_1 - w^*)^{-1} M (z_2 - w^*)^{-1}) \\ &\quad = \frac{\operatorname{tr} \tilde{M}(z_2 - w^*)}{\det(z_2 - w^*)} - \frac{\operatorname{tr} \tilde{M}(z_1 - w^*)}{\det(z_1 - w^*)}, \end{aligned} \tag{4.29}$$

where, for $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$, $\tilde{M} = \begin{pmatrix} m_4 & -m_2 \\ -m_3 & m_1 \end{pmatrix}$. We parametrize the elements k in $H(2)$ by

$$k = \begin{pmatrix} k_0 + k_1 & k_2 - ik_3 \\ k_2 + ik_3 & k_0 - k_1 \end{pmatrix}, \tag{4.30}$$

and we let

$$\frac{\partial}{\partial k} = \begin{pmatrix} \dfrac{\partial}{\partial k_0} + \dfrac{\partial}{\partial k_1} & \dfrac{\partial}{\partial k_2} - i\dfrac{\partial}{\partial k_3} \\ \dfrac{\partial}{\partial k_2} + i\dfrac{\partial}{\partial k_3} & \dfrac{\partial}{\partial k_0} - \dfrac{\partial}{\partial k_1} \end{pmatrix}. \tag{4.31}$$

Then the operator $(\operatorname{tr} \tilde{M}(\partial/\partial k))$ applied to k gives $2\tilde{M}$; applied to $\det k$ it gives $2 \operatorname{tr} Mk$, and applied to $\operatorname{tr} Mk$ it gives $2 \operatorname{tr} M\tilde{M} = 4 \det M$.

Let us now look at (4.28). According to (4.29) this breaks down into a sums of terms of the form

$$\frac{(\operatorname{tr} \tilde{M}(z_1 - w^*))^s}{\det(z_1 - w^*)^s} K_n'(z_1, w)\, v_1 \otimes \frac{(\operatorname{tr} \tilde{M}(z_2 - w^*))^{r-s}}{\det(z_2 - w^*)^{r-s}} K_m'(z_2, w)\, v_2 \,. \tag{4.22}$$

In this expression, for $s \geqslant 1$, and up to a constant,

$$\frac{1}{\det(z_1 - w^*)^s} K_n'(z_1, w)\, v_1 = \int_{C^+} \tau_n(k) \det k^{s-1} e^{i \operatorname{tr}(z_1 - w^*)k}\, dk v_1\,, \tag{4.23}$$

and $\operatorname{tr} \tilde{M}(z_1 - w^*)\, e^{i\operatorname{tr}(z_1 - w^*)k}$ is proportional to $(\operatorname{tr} \tilde{M}(\partial/\partial k))\, e^{i\operatorname{tr}(z_1-w^*)k}$.

From this and the preceding remarks it follows that if $e_1 = \binom{1}{0} \in \mathbb{C}^2$, if $M = \binom{1}{0}(0\ 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, if $v_1 = e_1 \otimes \cdots \otimes e_1$, and if $v_2 = e_1 \otimes \cdots \otimes e_1$, then the corresponding function (4.28) does indeed belong to $H_1(n, -1) \otimes H_1(m, -1)$. Thus, for $r \geqslant 1$, the representation $T_3(n + m + r, r, r + 2)$ is contained in the tensor product. To complete the analysis one could now either examine functions of the form (4.28) corresponding to the other highest weight vectors, or compare the K-types. We mention that the K-types for the mass-zero representations can be found by combining [7, pp. 100–104] with [12]. These possibilities are mentioned because the tensor product of a mass-zero representation with a representation supported by C^+ can be treated among these lines. However, for the given case, the intuitively clear result that besides what we pick up for $r = 0$, the above representations are the only representations that appear, follows readily from the remarks following Proposition 4.9 below. We mention that the case $T_1(n, -1) \otimes T_2(m, -1)$ and $T_2(n, -1) \otimes T_2(m, -1)$ are similar, and that the results also can be obtained from [8].

PROPOSITION 4.9.

$$T_1(n, -1) \otimes T_1(m, -1)$$
$$= \bigoplus_{q=0}^{\min(n,m)} T_1(n+m-2q, q) \oplus \bigoplus_{r=1}^{\infty} T_3(n+m+r, r, r+2);$$

$$T_2(n, -1) \otimes T_2(m, -1)$$
$$= \bigoplus_{s=0}^{\min(n,m)} T_2(n+m-2q, q) \oplus \bigoplus_{r=1}^{\infty} T_3(r, n+m+r, n+m+r+2);$$

$$T_1(n, -1) \otimes T_2(m, -1)$$
$$= \bigoplus_{r=1}^{\infty} T(n+r, m+r, m+r+2).$$

We conclude this article by observing that several interesting phenomena occur at the decomposition of the tensor product of a mass-zero representation with a mass-zero representation. One such is that most of the representations that appear in the decomposition themselves live in solution spaces to differential equations. The simples example of this is the tensor product of $U_{-1} = T_1(0, -1) = T_2(0, -1)$ with itself. Since U_{-1} is unitary in a space of solutions to $\Box\varphi = 0$, the Hilbert space in which $U_{-1} \otimes U_{-1}$ acts consists of functions that are solutions to $\Box_{z_1} f = \Box_{z_2} f = 0$. Under the change of variables (4.13) these equations can be combined into

$$(\Box_z + \Box_y) f = 0,$$

and

$$\left(\frac{\partial}{\partial z^0}\frac{\partial}{\partial y^0} - \frac{\partial}{\partial z^1}\frac{\partial}{\partial y^1} - \frac{\partial}{\partial z^2}\frac{\partial}{\partial y^2} - \frac{\partial}{\partial z^3}\frac{\partial}{\partial y^3}\right) f = 0, \tag{4.34}$$

where we have used the Pauli matrices as a basis of $gl(2, \mathbb{C})$.

Let us assume for simplicity that f is of the form $y_0 f_0 + y_1 f_1 + y_2 f_2 + y_3 f_3$. Then f under our method of decomposing tensor products is mapped into a function $h = (h_1, h_2, h_3, h_f)$ with $h_i(z) = f_i(z, z)$ for $i = 0, 1, 2, 3$, and hence, by 4.34), h satisfies the equation

$$\frac{\partial}{\partial z_0} h_0 - \frac{\partial}{\partial z_1} h_1 - \frac{\partial}{\partial z_2} h_2 - \frac{\partial}{\partial z_3} h_3 = 0. \tag{4.35}$$

If f is of the form $\sum_i p_i^r(y) f_i$ where the p_i^r's are homogeneous polynomials of degree, r, the resulting equation of course is more complicated.

The Eq. (4.35) is in nature a "gauge-condition." However, it may also indicate a "conserved current."

A similar phenomenon takes place for the tensor product of any two limits of the series (4.22) and this leads to the conclusion that with the possible exception of what is picked up for $r = 0$, the representations occurring in the decomposition cannot be strongly supported by C^+.

References

1. K. I. Gross and R. A. Kunze, Bessel functions and representation theory II. Holomorphic discrete series and metaplectic representations, *J. Functional Analysis* **25** (1977), 1–49.
2. K. I. Gross, W. J. Holman III, and R. A. Kunze, The generalized gamma function, new Hardy spaces, and representations of holomorphic type for the conformal group, *Bull. Amer. Math. Soc.* **83** (1977), 412–415.
3. Harish-Chandra, Representations of semisimple Lie groups IV, V, *Amer. J. Math.* **77** (1955), 743–777; **78** (1956), 1–41.
4. S. Helgason, "Differential Geometry and Symmetric Spaces," Academic Press, New York, 1962.
5. H. P. Jakobsen, Tensor products, reproducing kernels and power series, *J. Functional Analysis*, to appear.
6. H. P. Jakobsen, B. Ørsted, I. E. Segal, B. Speh, and M. Vergne, Symmetry and causality properties of physical fields, *Proc. Natl. Acad. Sci. USA* **75** (1978), 1609–1611.
7. H. P. Jakobsen and M. Vergne, Wave and Dirac operators and representations of the conformal group, *J. Functional Analysis* **24** (1977), 52–106.
8. M. Kashiwara and M. Vergne, On the Segal–Shale–Weil representations and harmonic polynomials, *Invent. Math.* **44** (1978).
9. B. Ørsted, Conformally invariant differential equations and projective geometry, *J. Functional Analysis*, to appear.
10. H. Rossi and M. Vergne, Analytic continuation of the holomorphic discrete series, *Acta Math.* **136** (1976), 1–59.
11. I. E. Segal, "Mathematical Cosmology and Extragalactic Astronomy," Academic Press, New York, 1976.
12. S. Sternberg and J. A. Wolf, Hermitian Lie algebras and metaplectic representations, to appear.
13. N. Wallach, On the unitarizability of representations with highest weights, *in* "Noncommutative Harmonic Analysis," Lecture Notes in Mathematics, No. 466, Springer, Berlin/Heidelberg/New York.
14. S. Martens, The characters of the holomorphic discrete series, *Proc. Nat. Acad. Sci. USA* **72** (1975), 3275–3276.